Generalizing Topology via Chu Spaces*

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Abstract

By using the representational power of Chu spaces we define the notion of a generalized topological space (or GTS, for short), i.e., a mathematical structure that generalizes the notion of a topological space. We demonstrate that these topological spaces have as special cases known topological spaces. Furthermore, we develop the various topological notions and concepts for GTS. Moreover, since the logic of Chu spaces is linear logic, we give an interpretation of most linear logic connectives as operators that yield topological spaces.

1 Introduction

Chu spaces are the objects of mathematics previously known as *games* [4], which are the result of a long evolution of the Chu construct, i.e., the enrichment of category $\mathbf{Chu}(V, k)$ over the category V. The Chu construct, which is named after Po-Hsiang Chu a student of Michael Barr, appears in the theory of *-autonomous categories (see [1] for a detailed description of the various related concepts and [2] for a recent account). The theory of Chu spaces has been developed by Pratt in an effort to provide an alternative representation of types and processes [7], and formal languages and finite state automata [6]. However, the representational power of Chu spaces is not limited to the above cases. For example Lafont and Streicher in [4] report that vector spaces, topologies and Girard's coherent spaces are among the mathematical entities that can be represented with Chu spaces; while Pratt in [5] reports that all "partially distributive lattice" categories can be realized by them. These remarks lead us to the definition of a generalization of the notion of a topological space which we call a *generalized topological space*, i.e., a topological space which, under certain conditions, has as special cases known topological spaces. We define all the usual notions associated with topological spaces, such as compactness, etc. Furthermore, since linear logic [3] is the logic of Chu spaces, we give an interpretation of most connectives of the logic as operators that yield new topological spaces.

2 Generalized Topological Spaces

We consider triplets of the form $(X, r, \mathfrak{A})$, where X is an arbitrary set called the *set of points*, $\mathfrak{A}$ is an arbitrary set called the *set of open sets*, and $r : X \times \mathfrak{A} \to I$ ($I = [0, 1]$) is an arbitrary binary function called the *membership function*. The intuitive meaning of this function is that for all $x \in X$ and $A \in \mathfrak{A}$, $r(x, A)$ is the degree to which x is a member of the open set A. Any such triplet will be called a *generalized topological space* (or GTS, for short).

Definition 2.1 A GTS $(X, r, \mathfrak{A})$ is called a *strong generalized topological space* (or SGTS, for short) if and only if the following two conditions are fulfilled:

[0]This is paper that was writen in 1999.

(i). If $A_1, A_2 \in \mathfrak{A}$, then there is an $A \in \mathfrak{A}$, such that

$$\min\Big\{r(x, A_1), r(x, A_2)\Big\} = r(x, A),$$

for all $x \in X$; we say that A is the *intersection* of A_1 and A_2, denoted as $A_1 \cap A_2$.

(ii). If $A_j \in \mathfrak{A}$, $j \in J$, there is an $A \in \mathfrak{A}$, such that

$$\sup\Big\{r(x, A_j), j \in J\Big\} = r(x, A),$$

for all $x \in X$; we say that A is the *infinite union* of A_j, denoted as $\bigcup_{j\in J} A_j$.

Note that we don't need to explicitly specify the elements of the union and the intersection. By imposing some restrictions on the range of values of r and/or the structure of $\mathfrak{A}$ we get common topological spaces:

Fuzzy Topology We assume that for a given SGTS, $\mathfrak{A} \subseteq I^X$, such that $\bar{\mathbf{0}}, \bar{\mathbf{1}} \in \mathfrak{A}$ ($\bar{\mathbf{0}}(x) = 0$ and $\bar{\mathbf{1}}(x) = 1$ for all $x \in X$) and so $r(x, A) = A(x)$, for all $A \in \mathfrak{A}$ and all $x \in X$. Then the two conditions become

(i). if $A_1, A_2 \in \mathfrak{A}$, then $A_1 \wedge A_2 \in \mathfrak{A}$,

(ii). if $\{A_j, j \in J\} \subseteq \mathfrak{A}$, then there is an $A \in \mathfrak{A}$, such that $\bigvee_{j\in J} A_j \in A$,

Then this is just the definition of a fuzzy topological space. We proceed now with the definition of some useful concepts.

Definition 2.2 Let $(X, r, \mathfrak{A})$ be a GTS and $A_1, A_2 \in \mathfrak{A}$, then $A_1 = A_2$ iff $r(x, A_1) = r(x, A_2)$ for all $x \in X$. Furthermore, we consider two GTS $(X, r, \mathfrak{A})$ and $(X, s, \mathfrak{B})$, any $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ are equal iff $r(x, A) = s(x, B)$, for all $x \in X$.

Definition 2.3 Let $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ be two GTSs, Then $\mathcal{B}$ is a *subspace* of $\mathcal{A}$, denoted as $\mathcal{B} \tilde{\subseteq} \mathcal{A}$, iff $Y \subseteq X$ and there is a surjection $\nu : \mathfrak{A} \to \mathfrak{B}$ such that $r(y, A) = s(y, \nu(A))$ for all $y \in Y$ and for all $A \in \mathfrak{A}$.

2.1 Closed Generalized Topological Spaces

Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a GTS and $\mathfrak{K}$ be an arbitrary set, such that there is a unique bijection $\varphi : \mathfrak{A} \to \mathfrak{K}$. Moreover, define a function $\bar{r} : X \times \mathfrak{K} \to I$, such that for all $K \in \mathfrak{K}$

$$\bar{r}(x, K) = 1 - r(x, \varphi^{-1}(K))$$

Then the GTS $\bar{\mathcal{A}} = (X, \bar{r}, \mathfrak{K})$ is called a *closed generalized topological space* (or CGTS, for short). We call $\mathfrak{K}$ the *set of closed sets*. Obviously, $r(x, A) = 1 - \bar{r}(x, \varphi(A))$. In case that $\mathfrak{A} \subseteq \mathcal{P}(X)$ for each $A \in \mathfrak{A}$, $\varphi(A) = A^{\complement}$ (i.e., $\varphi(A)$ is the *complement* of A); and for each $K \in \mathfrak{K}$, $\varphi^{-1}(K) = K^{\complement}$. In case that $\mathfrak{A} \subseteq I^X$, then from the definition of $\bar{r}$, we get that for all $K \in \mathfrak{K}$

$$\begin{aligned} K(x) &= \bar{r}(x, K) \\ &= 1 - r(x, \varphi^{-1}(K)) \\ &= 1 - r(x, K^{\complement}) \\ &= 1 - K^{\complement}(x) \end{aligned}$$

Which is just the definition of the complement in fuzzy topology. The following theorem proves that a SGTS and a bijection φ induce a CGTS with similar properties:

Theorem 2.1 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a SGTS and $\varphi : \mathfrak{A} \to \mathfrak{K}$ be a bijections such that $\bar{\mathcal{A}} = (X, \bar{r}, \mathfrak{K})$ is a CGTS. Then,*

(i). if $K_1, K_2 \in \mathfrak{K}$, there is a $K \in \mathfrak{K}$ such that

$$\max\Big\{\bar{r}(x, K_1), \bar{r}(x, K_2)\Big\} = \bar{r}(x, K), \quad \forall x \in X$$

and

(ii). if $K_j \in \mathfrak{K}$ and $j \in J$, there is a $K \in \mathfrak{K}$ such that

$$\inf\Big\{\bar{r}(x, K_j), j \in J\Big\} = \bar{r}(x, K), \quad \forall x \in X$$

Any CGTS which satisfies the two conditions of the previous theorem is called a *strong closed generalized topological space* (or SCGTS, for short).

Proposition 2.1 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a GTS and $\varphi_1 : \mathfrak{A} \to \mathfrak{K}$ be a bijection which induces the CGTS $\bar{\mathcal{A}} = (X, \bar{r}, \mathfrak{K})$. Moreover, let $\mathcal{B} = (Y, s, \mathfrak{B})$ be a subspace of $\mathcal{A}$ (i.e., among others there is a surjection $\nu_1 : \mathfrak{A} \to \mathfrak{B}$), and $\varphi_2 : \mathfrak{B} \to \mathfrak{L}$ be a bijection such that $\bar{\mathcal{B}} = (Y, \bar{s}, \mathfrak{L})$ is a CGTS. Then, $\bar{\mathcal{B}} \tilde{\subseteq} \bar{\mathcal{A}}$.*

2.2 Dual Generalized Topological Spaces

The dual of a GTS $\mathcal{A} = (X, r, \mathfrak{A})$, denoted as $\mathcal{A}^\perp$, is defined to be the triplet $(\mathfrak{A}, r^{\smile}, X)$, where $r^{\smile}(A, x)$ denotes the degree to which the open set $A \in \mathfrak{A}$ *contains* the point x, i.e., $r^{\smile}(A, x) = r(x, A)$. This means that the dual of a GTS and the GTS itself externally behave in the same way. Their only difference is their internal structure.

2.3 Subset-hood

Let $A_1, A_2 \in \mathfrak{A}$ be two open sets of a GTS $\mathcal{A} = (X, r, \mathfrak{A})$, and let $K_1, K_2 \in \mathfrak{K}$ be two closed sets of $\bar{\mathcal{A}}$, then we say that

- A_1 is a subset of A_2 (denoted $A_1 \subseteq A_2$) iff $r(x, A_1) \le r(x, A_2)$ for all $x \in X$,
- $K_1 \subseteq K_2$ iff $\bar{r}(x, K_1) \le \bar{r}(x, K_2)$ for all $x \in X$,
- A is a subset of of K iff $r(x, A) \le \bar{r}(x, K)$, for all $x \in X$, and
- K is a subset of of A iff $\bar{r}(x, K) \le r(x, A)$, for all $x \in X$.

Proposition 2.2 *Consider a GTS $\mathcal{A} = (X, r, \mathfrak{A})$, a bijection $\varphi : \mathfrak{A} \to \mathfrak{K}$, which indices the CGTS $\bar{\mathcal{A}}$, then for any $A_1, A_2 \in \mathfrak{A}$ such that $\varphi(K_1) = A_1$ and $\varphi(K_2) = A_2$:*

$$\begin{aligned} A_1 \subseteq A_2 &\quad\Leftrightarrow\quad \varphi(A_1) \supseteq \varphi(A_2) \\ K_1 \subseteq K_2 &\quad\Leftrightarrow\quad \varphi^{-1}(K_1) \supseteq \varphi^{-1}(K_2) \end{aligned}$$

2.4 Continuous Functions and Isomorphic Spaces

Continuous functions between two GTSs behave exactly like the morphisms of any category $\mathbf{Chu}(V, k)$, i.e., we consider two arbitrary GTS $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$; then a *continuous function* from $\mathcal{A}$ to $\mathcal{B}$ is a pair or functions $(f, \bar{f})$, where $f : X \to Y$ and $\bar{f} : \mathfrak{B} \to \mathfrak{A}$, such that for all $x \in X$ and $B \in \mathfrak{B}$ the following equation is true:

$$s(f(x), B) = r(x, \bar{f}(B))$$

Suppose that each function of the pair $(f, \bar{f})$ is a bijection, then we call the spaces $\mathcal{A}$ and $\mathcal{B}$ *isomorphic* to each other and we denote this by $\mathcal{A} \cong \mathcal{B}$. Moreover, this pair pair of functions is called an *isomorphism.*

Proposition 2.3 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ be two different GTSs. Moreover, the bijections $\varphi_1 : \mathfrak{A} \to \mathfrak{K}$ and $\varphi_2 : \mathfrak{B} \to \mathfrak{L}$ induce the CGTS $\bar{\mathcal{A}} = (X, \bar{r}, \mathfrak{K})$ and $\bar{\mathcal{B}} = (Y, \bar{s}, \mathfrak{L})$. Then the continuous transformation $(f, \bar{f})$ from $\mathcal{A}$ to $\mathcal{B}$, induces the continuous transformation $(f, \bar{f}^*)$ from $\bar{\mathcal{A}}$ to $\bar{\mathcal{B}}$, where $\bar{f}^* = \varphi_1 \circ \bar{f} \circ \varphi_2^{-1}$.*

The following result is a direct consequence of the previous proposition:

Proposition 2.4 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a GTS and $\varphi_1 : \mathfrak{A} \to \mathfrak{K}$ be a bijection such that $\bar{\mathcal{A}} = (X, \bar{r}, \mathfrak{K})$ is a CGTS. Similarly, let $\mathcal{B} = (X, r, \mathfrak{A})$ be a GTS, $\varphi_2 : \mathfrak{B} \to \mathfrak{L}$ be a bijection such that $\bar{\mathcal{B}} = (Y, \bar{s}, \mathfrak{L})$. Then,*

$$\bar{s}(f(x), L) = \bar{r}(x, \bar{f}^*(L)), \quad \forall x \in X, \forall L \in \mathfrak{L}$$

3 Compact Spaces

Since, *compactness* is a very useful topological concept we must provide a definition of it for our generalized topological spaces.

Definition 3.1 A GTS $\mathcal{A} = (X, r, \mathfrak{A})$ is *compact* iff for every family of open sets of $\mathfrak{A}$, i.e, $\{A_i, i \in I\}$, such that $\sup\{r(x, A_i), i \in I\} > 0$ for all $x \in X$, there is a finite subfamily, i.e., $\{A_j, j \in J\}$, where J is a finite subset of I, such that $\sup\{r(x, A_j), j \in J\} > 0$ for all $x \in X$.

Since, compactness is a property that must be preserved between isomorphic GTS, we prove the following proposition:

Proposition 3.1 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ be two different GTSs. If $\mathcal{A}$ is compact and $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{B}$ is also compact.*

4 Connected Spaces

We define the GTS $\mathbf{2} = (2, \phi, \mathcal{P}(2))$, where $2 = \{0, 1\}$ and $\phi(1, \{0\}) = 0$, $\phi(1, \{1\}) = 1$, etc.

Definition 4.1 Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a GTS. Then, $\mathcal{A}$ is called *connected* iff there is no continuous function $g : X \to 2$ which is surjective.

It is trivial to prove the following proposition:

Proposition 4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two GTS such that $\mathcal{A} \cong \mathcal{B}$. If $\mathcal{A}$ is connected, then $\mathcal{B}$ is also connected.*

5 Linear Implication—Pointwise Topologies

Let $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ be two GTS, then we define the *pointwise topological* space, denoted as $\mathcal{A} \multimap \mathcal{B}$, to be the GTS $(\mathcal{B}^{\mathcal{A}}, t, X \times \mathfrak{B})$, with $t(f, (x, B)) = s(f(x), B)$. A pointwise topology is nothing else than the function space of [6]. We now investigate the meaning of this topology in the classical and the fuzzy cases.

In the classical case the equation $t(f, (x, B)) = s(f(x), B)$ means that $f(x) \in B$ iff $f \in \langle x, B \rangle$, where $\langle x, B \rangle = \{g \in Y^X : g(x) \in B\}$, and the set $\{(x, B) : x \in X,\ B \in \mathcal{O}(Y)\}$ form a subbase in the function space $\mathcal{F}(X, Y)$. In the fuzzy setting we assume that $\langle x, B \rangle : \mathcal{F}(X, Y) \to I$, so $\langle x, B \rangle(f) = t(f, (x, B)) = B(f(x))$. This equation defines a topology which we call *fuzzy pointwise topology* on the set $\mathcal{F}(X, Y)$.

Consider the GTS $\mathcal{A} \multimap \mathcal{B} = (\mathcal{B}^{\mathcal{A}}, t, X \times \mathfrak{B})$ and the bijections $\varphi_2 : \mathfrak{B} \to \mathfrak{L}$ and $\varphi_* = \mathrm{id}_X \times \varphi_2$, which induce the CGTS $\bar{\mathcal{B}}$ and the CGTS:

$$\overline{\mathcal{A} \multimap \mathcal{B}} = (\mathcal{B}^{\mathcal{A}}, \bar{t}, X \times \mathfrak{L}),$$

respectively. It is easy to prove that $\bar{t}(f, (x, L)) = \bar{s}(f(x), L)$ for all $x \in X$, $L \in \mathfrak{L}$, and $f \in \mathcal{B}^{\mathcal{A}}$:

$$\begin{aligned}
\bar{t}(f, (x, L)) &= 1 - t(f, \varphi_*^{-1}(x, L)) \\
&= 1 - s(f(x), B) \\
&= \bar{s}(f(x), \varphi_2(B)) \\
&= \bar{s}(f(x), L)
\end{aligned}$$

Moreover, if $\varphi_1 : \mathfrak{A} \to \mathfrak{K}$ is a bijection, which induces the CGTS $\bar{\mathcal{A}}$, then the GTS $\bar{\mathcal{A}} \multimap \bar{\mathcal{B}}$ is defined to be the triplet: $(\bar{\mathcal{B}}^{\bar{\mathcal{A}}}, \tau, X \times \mathfrak{L})$, such that $\tau(f, (x, L)) = \bar{s}(f(x), L)$, for all $x \in X$ and $L \in \mathfrak{L}$. Furthermore, it is trivial to prove that $\overline{\mathcal{A} \multimap \mathcal{B}} \cong \bar{\mathcal{A}} \multimap \bar{\mathcal{B}}$.

6 Hausdorf Spaces and Regular Spaces

Definition 6.1 A GTS $\mathcal{A} = (X, r, \mathfrak{A})$ is called a *Hausdorf* space iff for every $x_1, x_2 \in X$, such that $x_1 \neq x_2$, there are $A_1, A_2 \in \mathfrak{A}$, such that $r(x_1, A_1) > 0$, $r(x_2, A_2) > 0$, and

$$\min\Big\{r(x, A_1), r(x, A_2)\Big\} = 0, \quad \forall x \in X.$$

It is trivial to prove that continuous functions preserve Hausdorf spaces. Moreover, the following result can be proved easily :

Proposition 6.1 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a GTS and $\mathcal{B} = (Y, s, \mathfrak{B})$ be a Hausdorf space, then the pointwise topological space $\mathcal{A} \multimap \mathcal{B}$ is a Hausdorf space.*

Definition 6.2 Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a GTS and $\bar{\mathcal{A}} = (X, \bar{r}, \mathfrak{K})$ be a CGTS such that $\varphi : \mathfrak{A} \to \mathfrak{K}$ is a bijection. Then, we call $\mathcal{A}$ a *regular* space iff for all $x \in X$ and all $K \in \mathfrak{K}$ there are $A_1, A_2 \in \mathfrak{A}$ such that $r(x, A_1) > 0$, $K \subseteq A_2$, i.e., $\bar{r}(x', K) \leq r(x', A_2)$ for all $x' \in X$, and the following holds

$$\min\Big\{r(x', A_1), r(x', A_2)\Big\} = 0, \quad \forall x' \in X$$

Regularity is preserved by isomorphisms:

Proposition 6.2 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a regular space and $\mathcal{B} = (Y, s, \mathfrak{B})$ a GTS, such that $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{B}$ is also regular.*

Proposition 6.3 *Let $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ be two GTSs. If $\mathcal{B}$ is regular, then $\mathcal{A} \multimap \mathcal{B}$ is also regular.*

7 The Tensor Product

We consider two GTSs $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$, then the *tensor product*, denoted as $\mathcal{A} \otimes \mathcal{B}$, is defined as follows:

$$\begin{aligned} \mathcal{A} \otimes \mathcal{B} &= (\mathcal{A} \multimap \mathcal{B}^{\perp})^{\perp} \\ &= (X \times Y, \breve{t}, (\mathcal{B}^{\perp})^{\mathcal{A}}) \end{aligned}$$

where $\breve{t}((x, y), f) = \breve{s}(f(x), y)$ and $f : X \to \mathfrak{B}$. The topological space $\mathcal{A} \otimes \mathcal{B}$ has as set of points the set $X \times Y$ and as set of open sets all the *induced* open sets f, i.e., $\breve{t}((x, y), f)$ denotes the degree to which (x, y) belongs to the *open set* f. This new GTS has a concrete interpretation in both ordinary and fuzzy topology. As usual we start with the interpretation in ordinary topology.

Every function $f : X \to \mathcal{O}(Y)$ induces on the Cartesian product $X \times Y$ a topology which has as subbase the set $\mathfrak{Y} = \{\langle f \rangle : \ f : X \to \mathcal{O}(Y)\}$, where $\langle f \rangle = \{(x, y) : y \in f(x)\}$. In the fuzzy case, every function $f : X \to \mathcal{F}(Y)$, induces a fuzzy subset of $X \times Y$, denotes as $\langle f \rangle$, which is defined as follows:

$$\langle f \rangle(x, y) = \Big(f(x)\Big)(y)$$

The set $\mathfrak{Y} = \{\langle f \rangle : \ f : X \to \mathcal{F}(Y)\}$ is a subbase for a fuzzy topology on the product $X \times Y$.

8 The Tensor Sum

Again, we consider two GTSs $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$, then the *tensor sum* is defined as follows:

$$\begin{aligned} \mathcal{A} \parr \mathcal{B} &= \mathcal{A}^{\perp} \multimap \mathcal{B} \\ &= (\mathcal{B}^{\mathcal{A}^{\perp}}, \tau, \mathfrak{A} \times \mathfrak{B}) \end{aligned}$$

where $\tau(f, (A, B)) = s(f(A), B)$ and $f : \mathfrak{A} \to Y$. Naturally, this defines a pointwise topology. In particular, in the case of ordinary topologies we have that $f : \mathcal{O}(X) \to Y$ and $f \in \langle A, B \rangle$ iff $f(A) \in B$. Similar conclusions can be derived for the fuzzy case (see section 5).

9 Topological Sum and Product

We define the *topological sum* and the *topological product* of any two GTS. Since, their definition make use of the concept of the *direct sum*, or just sum, of two sets A and B, denoted as $A + B$, we must say that $A + B = \{0\} \times A \cup \{1\} \times B$.

Definition 9.1 The topological sum of two GTS $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$, denoted as $\mathcal{A} \oplus \mathcal{B}$, is the triplet $(X + Y, t, \mathfrak{A} \times \mathfrak{B})$, where $t((x, y), (0, A)) = r(x, A)$ and $t((x, y), (1, B)) = s(y, B)$.

There is also a special GTS, $\mathsf{0} = (\emptyset, \alpha, \{1\})$, with the property that for any triplet $\mathcal{A} = (X, r, \mathfrak{A})$ the following relations hold:

$$\mathcal{A} \oplus \mathsf{0} \cong \mathcal{A} \cong \mathsf{0} \oplus \mathcal{A}$$

Definition 9.2 The topological product of two GTS $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$, denoted as $\mathcal{A} \,\&\, \mathcal{B}$, is the triplet $(X \times Y, t', \mathfrak{A} + \mathfrak{B})$, where $t((0, x), (A, B)) = r(x, A)$ and $t((1, y), (A, B)) = s(y, B)$.

The special GTS, $\top = (\{1\}, \alpha^{\smile}, \emptyset)$, has the property that for any triplet $\mathcal{A} = (X, r, \mathfrak{A})$ the following relations hold:

$$\mathcal{A} \,\&\, \top \cong \mathcal{A} \cong \top \,\&\, \mathcal{A}$$

It is easy to verify that $\top = \mathsf{0}^{\perp}$ and to prove that $\mathcal{A} \oplus \mathcal{B} \cong (\mathcal{A}^{\perp} \,\&\, \mathcal{B}^{\perp})^{\perp}$.

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